

A RATIO OF INTEGRATION BETWEEN GIT QUOTIENTS

ZACHARY MADDOCK

ABSTRACT. Any Chow class α on the geometric invariant theory quotient $X//G$ of a variety X by a reductive group G can be lifted to a Chow class $\tilde{\alpha}$ on the quotient $X//T$ by a split maximal torus. This paper studies the ratio between the integral of the class α on $X//G$ and the integral of the class $c_{\text{top}}(\mathfrak{g}/\mathfrak{t}) \smile \tilde{\alpha}$ on $X//T$, where $c_{\text{top}}(\mathfrak{g}/\mathfrak{t})$ denotes the top Chern class of the T -equivariant vector bundle on X induced by the quotient of the adjoint representation on the Lie algebra of G by that of T . We provide a characteristic-free proof that this ratio is an invariant of the group G , and we show that it equals the order of the Weyl group whenever the root system of G decomposes into irreducible root systems of type \mathbf{A}_n , for various $n \in \mathbb{N}$. We provide a purely algebraic proof of this latter fact, but also describe how to remove the restriction on root systems by applying a related result of Martin from symplectic geometry.

INTRODUCTION

The cohomology of quotients arising from geometric invariant theory (GIT) has been the object of much study. In [Ki1] Kirwan used the analyses of instability in the previous works of Hesselink [He], Kempf [Ke1], and Kempf-Ness [KN] to explore the structure of GIT quotients from both the algebraic and symplectic perspectives, finding formulas to compute Hodge numbers. Five years later, Ellingsrud and Strømme [ES] began a study of the relationship between the Chow rings of the two GIT quotients $\mathbb{P}^n//G$ and $\mathbb{P}^n//T$, for a reductive group G with maximal torus $T \subseteq G$, and used this to provide a presentation of the Chow ring $A^*(\mathbb{P}^n//G)_{\mathbb{Q}}$ in terms of explicit generators and relations. Brion [Br2] then expanded this relationship to smooth, projective varieties X over the complex numbers and proved that the G -equivariant cohomology of X_G^{ss} is isomorphic to the submodule of $H_T^*(X_T^{ss}; \mathbb{Q})$ comprising those classes anti-invariant under the action of the Weyl group:

$$(0.0.1) \quad \phi : H_G^*(X_G^{ss}; \mathbb{Q}) \cong H_T^*(X_T^{ss}; \mathbb{Q})^a.$$

Later Brion and Joshua [BJ] extended these results further to the case of singular X , but with equivariant *intersection* cohomology used as a suitable replacement for the standard theory.

Brion's construction of the isomorphism ϕ is as follows (cf. [Br2]). Since X_G^{ss} is a G -variety and $T \subseteq G$ is a subgroup, there is an injective homomorphism $\pi^* : H_G^*(X_G^{ss}; \mathbb{Q}) \rightarrow H_T^*(X_G^{ss}; \mathbb{Q})$ that induces an isomorphism onto the submodule of W -invariant elements $H_T^*(X_G^{ss}; \mathbb{Q})^W \subseteq H_T^*(X_G^{ss}; \mathbb{Q})$. Moreover, there is a W -equivariant isomorphism

$$(0.0.2) \quad H_T^*(X_G^{ss}; \mathbb{Q}) \cong S \otimes_{SW} H_G^*(X_G^{ss}; \mathbb{Q}),$$

where $S := H_T^*(\text{Spec } \mathbb{C}; \mathbb{Q})$ is the T -equivariant cohomology of a point, and under this identification π^* is equal to $1 \otimes \text{id}$. The open inclusion $i : X_G^{ss} \hookrightarrow X_T^{ss}$ induces a surjective homomorphism $i^* : H_T^*(X_T^{ss}; \mathbb{Q}) \rightarrow H_T^*(X_G^{ss}; \mathbb{Q})$, and Brion's key observation is that i^* is an isomorphism on the W -anti-invariant submodules:

$$i^* : H_T^*(X_T^{ss}; \mathbb{Q})^a \cong H_T^*(X_G^{ss}; \mathbb{Q})^a.$$

The anti-invariant elements $S^a \subseteq S$ form a free S^W module of rank 1, generated by the element defined as the product of the positive roots, $\sqrt{c_{\text{top}}} := \prod_{\alpha \in \Phi^+} \alpha \in S$; here S is identified with $\text{Sym}_{\mathbb{Q}}(\Lambda^*(T)_{\mathbb{Q}}) \cong S$, with $\Lambda^*(T)$ denoting the character group of T . Combining these facts, $\phi := (i^*)^{-1} \circ (\sqrt{c_{\text{top}}} \frown \pi^*)$ is the desired isomorphism. If $\tilde{\alpha} \in H_T^*(X_T^{ss}; \mathbb{Q})^W$ denotes some W -invariant lift of the class $\alpha \in H_G^*(X_G^{ss}; \mathbb{Q})$, that is, if $i^* \tilde{\alpha} = \pi^* \alpha$, then ϕ can be described explicitly as

$$\phi : \alpha \mapsto \sqrt{c_{\text{top}}} \frown \tilde{\alpha}.$$

In this paper we address the question of how this isomorphism ϕ interacts with the integration pairings on $X//G$ and $X//T$. When $X_G^{ss} = X_G^s$, there is a natural identification between the equivariant cohomology groups of the semi-stable locus and the ordinary cohomology groups of the GIT quotient,

$$H_G^*(X_G^{ss}; \mathbb{Q}) \cong H^*(X//G; \mathbb{Q}),$$

(and similarly with T replacing G). Hence, for any $\alpha_1, \alpha_2 \in H^*(X//G; \mathbb{Q})$, we can compare the integrals

$$\int_{X//G} \alpha_1 \frown \alpha_2 \stackrel{?}{\leftrightarrow} \int_{X//T} \phi(\alpha_1) \frown \phi(\alpha_2).$$

Because $\phi(\alpha_1) \frown \phi(\alpha_2) = (\sqrt{c_{\text{top}}} \frown \sqrt{c_{\text{top}}}) \frown (\tilde{\alpha}_1 \frown \tilde{\alpha}_2)$ and $i^*(\tilde{\alpha}_1 \frown \tilde{\alpha}_2) = \pi^*(\alpha_1 \frown \alpha_2)$, we may simplify notation by defining $\alpha := \alpha_1 \frown \alpha_2$. Moreover, it will prove to be more natural to consider the class $c_{\text{top}} := \prod_{\alpha \in \Phi} \alpha$ instead of $\sqrt{c_{\text{top}}} \frown \sqrt{c_{\text{top}}}$, which just differs from the former by the sign $(-1)^{|\Phi^+|}$. After these substitutions, the question becomes the comparison of the integrals $\int_{X//G} \alpha$ and $\int_{X//T} c_{\text{top}} \frown \tilde{\alpha}$ for $\alpha \in H^*(X//G; \mathbb{Q})$.

Martin answered this question in the context of symplectic geometry (cf. [Ma]). There he proved the following formula for Hamiltonian actions of compact Lie groups G on symplectic manifolds X :

$$(0.0.3) \quad \int_{X//G} \alpha = \frac{1}{|W|} \int_{X//T} c_{\text{top}} \frown \tilde{\alpha}.$$

Here $X//G$ and $X//T$ denote symplectic reductions. Furthermore, he used this to give a presentation of $H_G^*(X_G^{ss})$ as the quotient of the W -invariant elements of $H_T^*(X_T^{ss})$ by the elements annihilated by c_{top} :

$$H_G^*(X_G^{ss}) \cong H_T^*(X_T^{ss})^W / \text{Ann}(c_{\text{top}}).$$

Martin's method of proof is analytic, with the crux of his argument relying on properties of moment maps, while the methods in the works of Brion, Ellingsrud-Strømme, and Brion-Joshua mentioned above are algebraic.

The purpose of this note is to generalize Martin's results to the algebraic setting of varieties X over an arbitrary field k . Let G be a reductive group over k with a split maximal torus $T \subseteq G$ and Weyl group W . Let X be a G -linearized (possibly singular) variety over k for which $X_T^s = X_T^{ss} \neq \emptyset$. Denote by the symbol $\int_Y \sigma$ the degree of the Chow class $\sigma \in A_0(Y)_{\mathbb{Q}}$ of a proper variety $Y \rightarrow k$ given by proper push-forward, and by $c_{\text{top}} := \prod_{\alpha \in \Phi} \alpha \in A^*(BT)$ the top Chern class in the Chow ring of the vector bundle $\mathfrak{g}/\mathfrak{t}$ on the classifying space BT . We are led to define, for projective linearized G -varieties X , the *GIT integration ratio*:

$$r_{G,T}^{X,\alpha} := \frac{\int_{X//T} c_{\text{top}} \frown \tilde{\alpha}}{\int_{X//G} \alpha} \in \mathbb{Q},$$

for a Chow 0-cycle $\alpha \in A_0(X//G)_{\mathbb{Q}}$ that does not integrate to 0 and a lift $\tilde{\alpha} \in A_*(X//T)_{\mathbb{Q}}$ (cf. Defn. 1.2.2). Understanding the invariance properties of this ratio will guide us to the proper

generalization of Martin's theorem. The ratio $r_{G,T}^{X,\alpha}$ may depend on the choice of lift $\tilde{\alpha}$ or — as seems *a priori* more likely — the variety X , but neither is the case. This is our main theorem (proved in §3):

Theorem 0.0.4. *If G is a reductive group over a field k and $T \subseteq G$ a split maximal torus, then the GIT integration ratio $r_{G,T}^{X,\alpha}$, defined above for a G -linearized projective k -variety X and a Chow class $\alpha \in A_0(X//G)$ satisfying $X_T^{ss} = X_T^s$ and $\int_{X//G} \alpha \neq 0$, depends not on the choice of T , X , or α . That is, $r_G := r_{G,T}^{X,\alpha}$ is an invariant of the group G .*

The GIT integration ratio is multiplicative under the group operation of direct products and invariant under central extensions. This is the content of our second theorem (proved in §4):

Theorem 0.0.5. *If G is a reductive group over a field k that, up to central extensions, is the product of simple groups $G_1 \times \cdots \times G_n$, then*

$$r_G = \prod_{i=1}^n r_{G_i}.$$

As a result of these theorems, the determination of the value r_G for all reductive groups G is reduced to the computation of r_G just for the simple groups. We are able to do this explicitly in §5 for the simple group $G = PGL(n)$, where we verify $r_G = n! = |W|$.

Corollary 0.0.6. *Let G be a reductive group over a field k and $T \subseteq G$ a split maximal torus. If the root system of G decomposes into irreducible root systems of type \mathbf{A}_n , for various $n \in \mathbb{N}$, then for any G -linearized projective k -variety X for which $X_T^s = X_T^{ss}$ and any Chow class $\alpha \in A_0(X//G)_{\mathbb{Q}}$ with lift $\tilde{\alpha} \in A_*(X//T)_{\mathbb{Q}}$,*

$$(0.0.7) \quad \int_{X//G} \alpha = \frac{1}{|W|} \int_{X//T} c_{top} \frown \tilde{\alpha}.$$

For general reductive groups, one can apply Theorem 0.0.4 and make use of the theory of relative GIT (cf. [Se]) and specialization (cf. [Fu, §20.3]) to remove the root system condition from the above corollary by reducing the proof to a calculation over the complex numbers, where we may apply Martin's result (cf. [Ma, Thm. B]); this is discussed in §6. An entirely algebraic proof of the general case still eludes us.

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NOTATION

- k denotes a field and \bar{k} its algebraic closure.
- G denotes a smooth, reductive group over k , with identity element $e \in G$ and split maximal torus $T \subseteq G$.
- $N_T \subseteq G$ denotes the normalizer of T in G .
- $\Lambda^*(T)$ denotes the character group of T and $\Lambda_*(T)$ the group of 1-parameter subgroups.
- V denotes a finite dimensional G -representation over k .
- $\mathbb{P}(V)$ denotes the projective space of hyperplanes in V , so that $\Gamma(\mathbb{P}(V), \mathcal{O}(1)) = V$.
- \mathfrak{g} and \mathfrak{t} denote respectively the Lie algebras of G and T .
- W denotes the Weyl group of G .
- Φ (resp. Φ^+ or Φ^-) denotes the root system (resp. set of positive or negative roots) of G .

- X denotes a projective variety on which G acts and \mathcal{L} denotes an ample G -linearized line bundle.
- X_G^{ss} , X_G^s , and X_G^{un} denote the GIT loci of G -semi-stable, G -stable, and G -unstable points of X , respectively. We also use the analogous notations for T in place of G .
- $[S/H]$ denotes the quotient stack of a scheme S by a group H .
- $BT := [\mathrm{Spec} k/T]$ denotes the Artin stack that is the algebraic classifying space of T .
- $A_*(-)$ (resp. $A_*(-)_{\mathbb{Q}}$) denotes the Chow group with coefficients in \mathbb{Z} (resp. \mathbb{Q}), graded by dimension.
- $A^*(-)$ (resp. $A^*(-)_{\mathbb{Q}}$) denotes the operational Chow ring with coefficients in \mathbb{Z} (resp. \mathbb{Q}).
- $\sqrt{c_{\mathrm{top}}} := \prod_{\alpha \in \Phi^+} \alpha \in A^*(BT)$ and $c_{\mathrm{top}} := \prod_{\alpha \in \Phi} \alpha \in A^*(BT)$.
- $\mathcal{L} \boxtimes \mathcal{M}$ denotes the line bundle $\pi_Y^* \mathcal{L} \otimes \pi_Z^* \mathcal{M}$ on $Y \times Z$ when \mathcal{L} and \mathcal{M} are line bundles on varieties Y and Z , respectively.
- $\int_Y \alpha$ denotes the degree of a Chow class $\alpha \in A_0(Y)$ on a proper variety Y over k , computed via proper push-forward by the structure morphism.

1. ANALYSIS OF $\sqrt{c_{\mathrm{TOP}}}$ ON X_T^{ss} FOR SMOOTH X

The goal of this section is to prove that when X is smooth over an arbitrary field k , that the class $\sqrt{c_{\mathrm{top}}} \frown \tilde{\alpha}$ is independent of the choice of lift. From this the independence of $c_{\mathrm{top}} \frown \tilde{\alpha}$ follows immediately, as $c_{\mathrm{top}} = (-1)^{|\Phi^+|} \sqrt{c_{\mathrm{top}}} \frown \sqrt{c_{\mathrm{top}}}$. To treat the case of singular X , one could attempt a push-forward along a closed immersion into projective space. Unfortunately, complications arise due to the possible introduction of strictly semi-stable points; we postpone a discussion of these technicalities until §2.

1.1. A review of GIT. We give a brief summary of geometric invariant theory, mainly to set conventions. See [FKM] or [Ki1] for more detailed expositions.

Definition 1.1.1. Let X be a projective variety with a linearized action of a reductive group G , i.e. an ample line bundle $\pi : \mathcal{L} \rightarrow X$ and G -actions on \mathcal{L} and X for which π is an equivariant map. We call such an \mathcal{L} a G -linearization of X .

- ◊ The *semi-stable locus* is defined to be

$$X_G^{ss} := \{x \in X : \exists n > 0 \text{ and some } \phi \in \Gamma(X, \mathcal{L}^{\otimes n})^G \text{ satisfying } \phi(x) \neq 0\};$$

- ◊ The *stable*¹ *locus* is defined to be

$$X_G^s := \{x \in X_G^{ss} : G \cdot x \subseteq X_G^{ss} \text{ is a closed subscheme and } |\mathrm{stab}_G x| < \infty\}.$$

- ◊ The *unstable locus* is defined to be $X_G^{un} := X \setminus X_G^{ss}$.
- ◊ The *strictly semi-stable locus* is defined to be $X_G^{sss} := X_G^{ss} \setminus X_G^s$.

The following theorem justifies the making of the above definitions:

Theorem 1.1.2 (Mumford). *The open locus $X_G^{ss} \subseteq X$ of semi-stable points admits a uniform categorical quotient $\pi : X_G^{ss} \rightarrow X//G$, called the GIT quotient of X by G . Moreover, the ample line bundle \mathcal{L} descends to an ample line bundle on the projective scheme $X//G$, and the restriction $\pi|_{X_G^s} : X_G^s \rightarrow \pi(X_G^s)$ of π to the open stable locus is a geometric quotient.*

Proof. See [FKM, Thm. 1.10]. ■

¹Often in the literature, this is referred to as “properly stable”.

We conclude this review by describing how to compute the stable and semi-stable loci in practice. Let $T \subseteq G$ be a split maximal torus with character group $\Lambda^*(T)$. Equivariantly embed X into $\mathbb{P}(V)$ for some G -representation V ; e.g. via some tensor power of the G -linearized line bundle \mathcal{L} , with $V := \Gamma(X, \mathcal{L}^{\otimes n})$. The G -representation structure on V endows $\mathbb{P}(V)$ with a naturally linearized G -action, for which $\mathbb{P}(V)_G^{ss} \cap X = X_G^{ss}$ (and similarly for the stable loci). As a T -module, V decomposes as the direct sum of weight spaces $V = \bigoplus_{\chi \in \Lambda^*(T)} V_\chi$.

Definition 1.1.3. For any $x \in X \subseteq \mathbb{P}(V)$ as above, we define the *state* of x to be

$$\Xi(x) := \{\chi \in \Lambda^*(T) : \exists v \in V_\chi \text{ such that } v(x) \neq 0\}.$$

Theorem 1.1.4 (Hilbert-Mumford criterion). *A point x is semi-stable for the induced linearized T -action if and only if 0 is in the convex hull of $\Xi(x)$ in $\Lambda^*(T) \otimes \mathbb{Q}$. Moreover, x is stable for the T -action if and only if 0 is in the interior of the convex hull of $\Xi(x)$. Furthermore,*

$$X_G^{ss} = \bigcap_{g \in G} g \cdot X_T^{ss}, \text{ and } X_G^s = \bigcap_{g \in G} g \cdot X_T^s.$$

Proof. See [FKM, Thm. 2.1]. ■

1.2. Lifting Chow classes between GIT quotients. We now define what it precisely means to lift a Chow class $\alpha \in A_*(X//G)_\mathbb{Q}$ to a class $\tilde{\alpha} \in A_*(X//T)_\mathbb{Q}$. We make use of the notion of Chow groups of quotient stacks, a review of which may be found in the appendix of this paper. The main result one needs to recall is that when $X_G^{ss} = X_G^s$, the quotient $[X_G^{ss}/G]$ is a proper Deligne-Mumford stack with coarse moduli space $\phi^G : [X_G^{ss}/G] \rightarrow X//G$, and there is an induced isomorphism

$$(1.2.1) \quad \phi_*^G : A_*([X_G^{ss}/G])_\mathbb{Q} \cong A_*(X//G)_\mathbb{Q}.$$

The upshot is that via the identification ϕ_*^G , we may think of a Chow class $\alpha \in A_*(X//G)_\mathbb{Q}$ equivalently as a Chow class $\alpha \in A_*([X_G^{ss}/G])_\mathbb{Q}$.

Definition 1.2.2. Let $\alpha \in A_*([X_G^{ss}/G])$. We say that the class $\tilde{\alpha} \in A_{*+g-t}([X_T^{ss}/T])$ is a *lift* of α provided that $i^*(\tilde{\alpha}) = \pi^*(\alpha)$, where $g := \dim G$, $t := \dim T$,

- ◇ $i : [X_G^{ss}/T] \hookrightarrow [X_T^{ss}/T]$ is the open immersion, and
- ◇ $\pi : [X_G^{ss}/T] \rightarrow [X_G^{ss}/G]$ is the flat fibration with fibre G/T .

Furthermore, we say $\tilde{\alpha} \in A_*(X//T)_\mathbb{Q}$ is a *lift* of $\alpha \in A_*(X//G)_\mathbb{Q}$ provided that $\phi_*^T(\tilde{\alpha})$ is a lift, in the above sense, of the Chow class $\phi_*^G(\alpha)$.

Remark 1.2.3. By the right exact sequence of Chow groups

$$A_*([X_G^{un} \cap X_T^{ss}/T]) \rightarrow A_*([X_T^{ss}/T]) \rightarrow A_*(X_G^{ss}/T) \rightarrow 0,$$

any two lifts of α differ by the push-forward of an element of $A_*([X_G^{un} \cap X_T^{ss}/T])$.

1.3. $\sqrt{c_{\text{top}}}$ is zero on each Kirwan stratum. We pause a moment to clarify the definition of $\sqrt{c_{\text{top}}}$. Recall that $A^*(BT) \cong \text{Sym}^*(\Lambda^*(T))$, where $\Lambda^*(T)$ is the character group of T , hence any polynomial in the roots $\alpha \in \Phi$ can be viewed as a Chow class in $A^*(BT)$. Moreover, for any T -variety Y (e.g. $Y = X_T^{ss}$), there is a flat morphism from the quotient stack $[Y/T]$ to the classifying space BT , so one may pull-back classes from $A^*(BT)$ to $A_*([Y/T])$. In this way, $\sqrt{c_{\text{top}}}$ defines a Chow class in $A_*([Y/T])$ for any T -variety Y .

In light of Remark 1.2.3, to show that $\sqrt{c_{\text{top}}} \frown \tilde{\alpha}$ is independent of the choice of lift, it suffices to show that $\sqrt{c_{\text{top}}}$ kills all elements in the image of $A_*([X_G^{un} \cap X_T^{ss}/T])$. This will be accomplished by the end of §1. In this subsection, we assume that X is a smooth, G -linearized projective variety over $k = \bar{k}$, and we decompose the locus $X_G^{un} \cap X_T^{ss}$ into strata on which to analyze $\sqrt{c_{\text{top}}}$. The

stratification we use is due to Kirwan, but relies on the previous work of Hesselink, Kempf, and Ness (cf. [He], [Ke1], [KN]). We summarize the relevant properties:

Theorem 1.3.1 (Kirwan). *Let X be a G -linearized projective variety over an algebraically closed field \bar{k} , with $T \subseteq G$ a choice of maximal torus. The unstable locus X_G^{un} admits a finite G -equivariant stratification*

$$X_G^{un} = \bigcup_{\beta \in \mathbf{B}} S_\beta$$

with the following properties:

- (1) $S_\beta \subseteq X_G^{un}$ is a locally closed G -equivariant subscheme.
- (2) $S_\beta \cap S_{\beta'} = \emptyset$ for $\beta \neq \beta'$.
- (3) There exist parabolic subgroups $P_\beta \subseteq G$, containing T , and locally closed P_β -closed subschemes $Y_\beta \subseteq S_\beta \cap X_T^{un}$ and a surjective G -equivariant morphism

$$\phi : Y_\beta \times_{P_\beta} G \rightarrow S_\beta$$

induced by the multiplication morphism $Y_\beta \times G \rightarrow S_\beta$.

If moreover X is smooth, then each S_β is smooth and the morphism ϕ in (3) is an isomorphism.

Proof. See [Ki1, §13]. ■

As a step toward showing that $\sqrt{c_{\text{top}}}$ vanishes on $X_G^{un} \cap X_T^{ss}$, we prove that $\sqrt{c_{\text{top}}}$ vanishes on each individual stratum $S_\beta \cap X_T^{ss}$. Since X is smooth, the stratum S_β is fibred over a flag variety G/P_β , so we first study $\sqrt{c_{\text{top}}}$ on G/P_β . To do so, we require some notation related to elements of the Weyl group. For a parabolic subgroup $P \subseteq G$, denote by W_P the subgroup $(N_T \cap P)/T \subseteq W$. Let the symbol \bar{w} denote a choice of a lift to N_T of an element $w \in W = N_T/T$, and let the symbol \bar{w} denote the image of w in W/W_P .

Lemma 1.3.2. *Let $P \subseteq G$ be a parabolic subgroup containing the maximal torus T , and let $i : W/W_P \rightarrow G/P$ be the inclusion defined by $\bar{w} \mapsto \bar{w}P$. Then i is simply the inclusion of the T -fixed points of G/P , and the Gysin pull-back of the T -equivariant class $[eP] \in A_*^T(G/P)$ is given by*

$$i^*([eP]) = \left(\prod_{\alpha \in \Phi(\mathfrak{g}/\mathfrak{p})} \alpha \right) \cdot [\bar{e}] \in A_*^T(W/W_P),$$

where $A_*^T(-)$ denotes the T -equivariant Chow group and $\Phi(\mathfrak{g}/\mathfrak{p})$ is the subset of roots consisting of the weights corresponding to the T -action on the tangent space $T_{eP}(G/P)$.

Proof. It is well-known that the T -invariant points of G/P are precisely W/W_P . The Chow group $A_*^T(W/W_P)$ is a free $A^*(BT)$ -module with basis given by the elements of W/W_P . The element $eP \in G/P$ is an isolated, nonsingular fixed point, disjoint from all other fixed points $wP \neq eP$. Hence, $i^*([eP])$ equals the self-intersection of $[eP]$. This equals the product of $[\bar{e}]$ and the T -equivariant top Chern class of the normal bundle $T_{eP}(G/P)$, which is clearly the product of the roots in $\Phi(\mathfrak{g}/\mathfrak{p})$. ■

Lemma 1.3.3. *Let $P \subseteq G$ be a parabolic subgroup containing the maximal torus T , let $\Phi(\mathfrak{g}/\mathfrak{p})$ be the collection of weights of the induced T -action on the tangent space $T_{eP}(G/P)$, and let $U \subseteq G/P$ denote the open complement of the finite set $W/W_P \hookrightarrow G/P$. As an element of the T -equivariant operational Chow group,*

$$\sqrt{c_{\text{top}}} = 0 \in A_T^*(U).$$

Proof. The variety U is smooth, so by Poincaré duality (Theorem A.2.3) it suffices to prove that $\sqrt{c_{\text{top}}} \frown [U] = 0 \in A_*^T(U)$. Let $X := G/P$ denote the flag variety, and let $i : X^T = W/W_P \rightarrow X$ denote the inclusion of the T -fixed points. By the right-exact sequence of Chow groups

$$A_*^T(X^T) \xrightarrow{i_*} A_*^T(X) \rightarrow A_*^T(U) \rightarrow 0,$$

it suffices to show that $\sqrt{c_{\text{top}}} \frown [X]$ is in the image of i_* . X is a smooth projective variety, so by the localization theorem (Theorem A.1.2) there is an injective $A^*(BT)$ -algebra homomorphism,

$$i^* : A_T^*(X) \rightarrow A_T^*(X^T).$$

Thus, it suffices to prove that $i^*(\sqrt{c_{\text{top}}} \frown [X])$ is in the image of $i^* \circ i_*$.

The ring $A^*(X^T)$ is a free $A^*(BT)$ -module with basis given by $\{[\bar{w}] : \bar{w} \in W/W_P\}$, with multiplication defined by the rule

$$[\bar{w}] \cdot [\bar{w}'] = \begin{cases} [\bar{w}] & : \text{if } \bar{w} = \bar{w}' \in W/W_P \\ 0 & : \text{otherwise,} \end{cases}$$

and in terms of this basis,

$$i^*(\sqrt{c_{\text{top}}} \frown [X]) = \sum_{\bar{w} \in W/W_P} \sqrt{c_{\text{top}}} \cdot [\bar{w}].$$

By Lemma 1.3.2,

$$i^* \circ i_*([\bar{e}]) = \left(\prod_{\alpha \in \Phi(\mathfrak{g}/\mathfrak{p})} \alpha \right) \cdot [\bar{e}].$$

Since i is a W -equivariant inclusion, the homomorphisms i^* and i_* are also compatible with W -action, hence

$$i^* \circ i_*([\bar{w}]) = \left(\prod_{\alpha \in \Phi(\mathfrak{g}/\mathfrak{p})} w\alpha \right) \cdot [\bar{w}].$$

Notice that $\Phi(\mathfrak{g}/\mathfrak{p})$ is a subset of the negative roots Φ^- , so for $\beta_w := \prod_{\alpha \in \Phi^- \setminus \Phi(\mathfrak{g}/\mathfrak{p})} w\alpha$,

$$\begin{aligned} i^* \circ i_*(\beta_w \cdot [wP/P]) &= \left(\prod_{\alpha \in \Phi^-} w\alpha \right) \cdot [\bar{w}] \\ &= \det(w) \cdot \sqrt{c_{\text{top}}} \cdot [\bar{w}]. \end{aligned}$$

Therefore, $\sum \sqrt{c_{\text{top}}} \cdot [\bar{w}] = i^* \circ i_*(\sum \det(w) \cdot \beta_w \cdot [\bar{w}])$. ■

We conclude the subsection by proving that $\sqrt{c_{\text{top}}}$ is zero on each stratum $S_\beta \cap X_T^{ss}$.

Lemma 1.3.4. *The Chow class $\sqrt{c_{\text{top}}}$ is zero as an element of the T -equivariant operational Chow group*

$$\sqrt{c_{\text{top}}} = 0 \in A_T^*(S_\beta \cap X_T^{ss}).$$

Proof. Since $S_\beta \cap X_T^{ss}$ is smooth, by Poincaré duality (Thm. A.2.3) it is enough to show that $\sqrt{c_{\text{top}}} \frown [S_\beta \cap X_T^{ss}] = 0 \in A_*^T(S_\beta \cap X_T^{ss})$. By Theorem 1.3.1, there is G -equivariant morphism $\pi : S_\beta \rightarrow G/P$ with $\pi^{-1}(eP) = Y_\beta \subseteq X_T^{un}$. Moreover, for any element $\dot{w} \in N_T$, we still have $\dot{w}Y_\beta \subseteq X_T^{un}$. By the G -equivariance, we therefore have the restriction of π satisfying $S_\beta \cap X_T^{ss} \rightarrow G/P - WP/P$. We finish by noting that $\sqrt{c_{\text{top}}}$ on $S_\beta \cap X_T^{ss}$ is the pull-back of the class $\sqrt{c_{\text{top}}}$ in $A_T^*(G/P - WP/P)$, which is 0 by Lemma 1.3.3. ■

Remark 1.3.5. *The arguments above in §1.3 are directly analogous to those used by Brion in [Br2] for equivariant cohomology, but the arguments to follow in §1.4 and §1.5 are original and yield marginally stronger results than what can be found in the literature for equivariant cohomology: when possible we use \mathbb{Z} -coefficients instead of \mathbb{Q} -coefficients.*

1.4. $\sqrt{c_{\text{top}}}$ is zero on $X_G^{\text{un}} \cap X_T^{ss}$. We continue to assume that X is smooth over an algebraically closed field \bar{k} , and we extend the vanishing of $\sqrt{c_{\text{top}}}$ on a stratum to vanishing over the entire locus $X_G^{\text{un}} \cap X_T^{ss}$, proving $\sqrt{c_{\text{top}}}$ acts as 0 on $A_*^T(X_G^{\text{un}} \cap X_T^{ss})_{\mathbb{Z}}$.

We recall a presentation of T -equivariant Chow groups given by Brion:

Proposition 1.4.1 (Brion). *Let X be a T -scheme. The T -equivariant Chow group $A_*^T(X)$ is generated as an $A^*(BT)$ -module by the classes $[Y]$ associated to T -invariant closed subschemes $Y \hookrightarrow X$.*

Proof. See [Br1, Thm. 2.1]. ■

Our first step is to extend the vanishing of $\sqrt{c_{\text{top}}}$ to the closure of each Kirwan stratum in $X_G^{\text{un}} \cap X_T^{ss}$. From this, the result on the entire space follows quickly.

Lemma 1.4.2. *Let \overline{S}_β be the closure of an unstable Kirwan stratum. The Chow class $\sqrt{c_{\text{top}}}$ annihilates every class in $A_*^T(\overline{S}_\beta \cap X_T^{ss})_{\mathbb{Z}}$.*

Proof. We proceed by induction on the dimension of the strata S_β . The result is clear for closed strata S_β by Lemma 1.3.4. Assume that S_β is a stratum and that all strata in its closure satisfy the conclusion of the lemma. By Proposition 1.4.1, it suffices to show that $\sqrt{c_{\text{top}}} \cap [Y] = 0 \in A_*^T(\overline{S}_\beta \cap X_T^{ss})$ for any T -invariant subvariety $Y \hookrightarrow \overline{S}_\beta \cap X_T^{ss}$. If $Y \cap S_\beta = 0$, then Y is contained in some $\overline{S}_{\beta'} \cap X_T^{ss}$ for a stratum $S_{\beta'}$ in the closure of S_β , and therefore is killed by the operator $\sqrt{c_{\text{top}}}$, as implied by the inductive hypothesis. Thus, the only case we need to consider is when Y intersects S_β nontrivially. We now resolve the birational map $\overline{S}_\beta \dashrightarrow G/P_\beta$ defined on S_β by $S_\beta \cong Y_\beta \times_{P_\beta} G \rightarrow G/P_\beta$ (cf. Theorem 1.3.1). Our strategy will be to partially resolve the locus of indeterminacy in the following manner:

$$\begin{array}{ccc} G \times_{P_\beta} \overline{Y}_\beta & & \\ \pi \downarrow & \searrow \tilde{f} & \\ \overline{S}_\beta & \dashrightarrow^f & G/P_\beta. \end{array}$$

The morphism π is a proper, since G/P_β is projective, and an isomorphism restricted to the dense open $S_\beta \subseteq \overline{S}_\beta$, by Theorem 1.3.1(3). The morphism \tilde{f} is a \overline{Y}_β -fibration, hence flat, and both morphisms \tilde{f} and π are T -equivariant. Since the morphisms are G -equivariant, and N_T preserves X_T^{un} , the above diagram factors as

$$\begin{array}{ccc} \pi^{-1}(\overline{S}_\beta \cap X_T^{ss}) & & \\ \pi \downarrow & \searrow \tilde{f} & \\ \overline{S}_\beta \cap X_T^{ss} & \dashrightarrow^f & U, \end{array}$$

where U is the open complement of the fixed-point locus, $U := G/P_\beta - WP_\beta/P_\beta \subseteq G/P_\beta$. If \tilde{Y} denotes the strict transform of Y under the birational morphism π , then the projection formula implies $\pi_*(\sqrt{c_{\text{top}}} \cap [\tilde{Y}]) = \sqrt{c_{\text{top}}} \cap [Y]$. The former equals 0, since $\sqrt{c_{\text{top}}}$ is the pull-back via \tilde{f}

of $\sqrt{c_{\text{top}}}$ from $(G/P_\beta) \setminus (WP_\beta/P_\beta)$, and this is 0 as an element of the operational Chow groups by Lemma 1.3.3. Therefore, $\sqrt{c_{\text{top}}} \frown [Y] = 0$, as desired. \blacksquare

Proposition 1.4.3. *Let X be a smooth G -linearized projective variety over \bar{k} . The Chow class $\sqrt{c_{\text{top}}}$ annihilates every class in $A_*^T(X_G^{un} \cap X_T^{ss})_{\mathbb{Z}}$.*

Proof. Since the stratification in Theorem 1.3.1 is finite, any closed (T -invariant) subscheme $Y \subseteq X_G^{un}$ must be contained in some $\overline{S_\beta}$, so the projection formula and Lemma 1.4.2 guarantee $\sqrt{c_{\text{top}}} \frown [Y] = 0$. The result then follows from Proposition 1.4.1. \blacksquare

1.5. $\sqrt{c_{\text{top}}}$ vanishes for \mathbb{Q} -coefficients when $k \neq \bar{k}$. We relax our previous assumption on k , allowing k to be an arbitrary field. Theorem 1.3.1 is proven over algebraically closed field, so our previous arguments do not immediately apply. If we weaken our statements by ignoring torsion, considering only Chow groups with rational coefficients, we can easily extend our previous results to this case. We outline the proof of the following well-established lemma (cf. [Bl, Lem. 1A.3]) for the reader's convenience.

Lemma 1.5.1. *If X is a variety over a field k , then any field extension K/k induces an injective morphism between Chow groups with rational coefficients: $A_*(X)_{\mathbb{Q}} \hookrightarrow A_*(X_K)_{\mathbb{Q}}$.*

Proof. If a field extension E/k is the union of a directed system of sub-extensions E_i/k , then $A_*(X_E) = \varinjlim A_*(X_{E_i})$. We may apply this result to the extension K/k to reduce the proof to the two cases: K/k is finite; or $K = k(x)$ for a transcendental element x .

If K/k is finite then the morphism $\phi : X_K \rightarrow X$ is proper and the composition $\phi_* \circ \phi^*$ is simply multiplication by $[K : k]$, which is an isomorphism since coefficients are rational. Therefore, ϕ^* is injective.

If $K = k(x)$, then X_K is the generic fibre of $\pi : X \times \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$. Let $\phi : X \times \mathbb{P}_k^1 \rightarrow X$ denote the other projection. There is an isomorphism of Chow groups $A_i(X \times \mathbb{P}_k^1)_{\mathbb{Q}} \cong A_{i-1}(X)_{\mathbb{Q}} \oplus A_i(X)_{\mathbb{Q}} \cdot t$, where t is the class associated to a fibre of π and the morphism $\phi^* : A_i(X)_{\mathbb{Q}} \rightarrow A_{i+1}(X \times \mathbb{P}_k^1)_{\mathbb{Q}}$ is identified with $\text{id} \oplus 0$ (cf. [Fu, Thm. 3.3]). As schemes,

$$X_K = \varinjlim_{\emptyset \neq U \subseteq \mathbb{P}^1} X \times U,$$

and there is an induced isomorphism on the level of Chow groups:

$$A_i(X_K)_{\mathbb{Q}} \cong \varinjlim_{\emptyset \neq U \subseteq \mathbb{P}^1} A_{i+1}(X \times U)_{\mathbb{Q}} \cong A_i(X)_{\mathbb{Q}} \oplus 0.$$

From this description, it is clear that the induced morphism $\phi^* : A_*(X)_{\mathbb{Q}} \rightarrow A_*(X_K)_{\mathbb{Q}}$ is an isomorphism, ergo injective. \blacksquare

Proposition 1.5.2. *Let X be a G -linearized smooth projective variety over k . Then $\sqrt{c_{\text{top}}}$ annihilates every class in $A_*^T(X_G^{un} \cap X_T^{ss})_{\mathbb{Q}}$.*

Proof. Lemma 1.5.1 reduces to the algebraically closed case, which is proved in Proposition 1.4.3. \blacksquare

As a corollary, we obtain

Corollary 1.5.3. *Let X be a smooth G -linearized projective variety over a field k so that $X_T^{ss} = X_T^s$. If $\tilde{\alpha}_0, \tilde{\alpha}_1 \in A_*([X_T^{ss}/T])_{\mathbb{Q}}$ are two lifts of $\alpha \in A_*([X_G^{ss}/G])_{\mathbb{Q}}$, then*

$$\int_{X//T} \sqrt{c_{\text{top}}} \frown \tilde{\alpha}_0 = \int_{X//T} \sqrt{c_{\text{top}}} \frown \tilde{\alpha}_1.$$

2. DEALING WITH STRICTLY SEMI-STABLE POINTS

One approach to extend the results of §1 to the case of singular X is to study a G -equivariant closed immersion $j : X \hookrightarrow \mathbb{P}(V)$ associated to the linearization. Regrettably, I see no reason to believe that the push-forward map $j_* : A_*([X_T^{ss}/T])_{\mathbb{Q}} \rightarrow A_*([\mathbb{P}(V)_T^{ss}/T])_{\mathbb{Q}}$ is injective, thus preventing an easy reduction of the proof for X to the known case of $\mathbb{P}(V)$ (cf. Prop. 1.5.2).

Our solution is to restrict to the case $X_T^{ss} = X_T^s$ and to limit our ambitions to showing that for any $\alpha \in A_*(X//G)_{\mathbb{Q}}$, the Chow class $\sqrt{c_{\text{top}}} \frown \tilde{\alpha}$ has a well-defined *numerical* equivalence class, independent of the choice of lift $\tilde{\alpha}$. Numerical equivalence would then behave better than algebraic equivalence under closed immersions, if not for the unfortunate possibility that $\mathbb{P}(V)_T^{ss} \neq \mathbb{P}(V)_T^s$, as there is not a well-defined notion of numerical equivalence on the Artin stack $[(\mathbb{P}(V))_T^{ss}/T]$. We circumvent this problem by building an auxiliary smooth G -linearized variety $Y \rightarrow \mathbb{P}(V)$, for which $Y_T^{ss} = Y_T^s$ and $Y_G^{ss} = Y_G^s \neq \emptyset$, and then relating integration on $X//G$ and $X//T$ to integration on $Y//G$ and $Y//T$. If one is guaranteed G -equivariant resolutions of singularities (e.g. if $\text{char } k = 0$), then a result of Reichstein [Re] generalizes the partial desingularizations of Kirwan [Ki2] and produces such an auxiliary variety Y . As resolution of singularities is still an open area of research in positive characteristic, we provide an independent construction. We will find this construction again useful in §3, when the existence of strictly semi-stable points would otherwise impede our efforts to show that the GIT integration ratio $r_{G,T}^{X,\alpha}$ is independent of X .

As a corollary of our method, we prove in §2.3:

Corollary 2.0.1. *Let G be a reductive group over k , $T \subseteq G$ a split maximal torus, and X a G -linearized projective variety X satisfying $X_T^{ss} = X_T^s$. If $\tilde{\alpha}_0, \tilde{\alpha}_1 \in A_*([X_T^{ss}/T])_{\mathbb{Q}}$ are two lifts of $\alpha \in A_*([X_G^{ss}/G])_{\mathbb{Q}}$, then*

$$\int_{X//T} \sqrt{c_{\text{top}}} \frown \tilde{\alpha}_0 = \int_{X//T} \sqrt{c_{\text{top}}} \frown \tilde{\alpha}_1.$$

2.1. The existence of semi-stable points. To construct the auxiliary space Y , we must understand the stabilizers of strictly semi-stable points $x \in X_T^{ss}$. We review some of their key properties in the next set of lemmas.

Lemma 2.1.1. *Let X be a T -linearized projective variety. If $X_T^{ss} \neq X_T^s$, then there exists a strictly semi-stable $x \in X_T^{ss}$ that has a positive dimensional stabilizer.*

Proof. If $X_T^{ss} \neq X_T^s$, then by definition there is a point with positive dimensional stabilizers, or there is a non-closed orbit in X_T^{ss} . The closure of a non-closed orbit contains an orbit of strictly smaller dimension, which then must have a positive dimensional stabilizer. \blacksquare

Lemma 2.1.2. *Let $(X, \mathcal{O}_X(1))$ be a projective variety T -linearized by a very ample line bundle $\mathcal{O}_X(1)$. If $x \in X_T^{ss}$ is strictly semi-stable and $T' \subseteq T$ is a subtorus stabilizing x , then T' acts trivially on the fibre $\mathcal{O}_X(1)|_x$.*

Proof. By the Hilbert-Mumford criterion, x must also be T' semi-stable. Since T' fixes x , the T' -state of x can only consist of a single weight. By the Hilbert-Mumford criterion, this weight must be 0, so T' acts trivially on the fibre $\mathcal{O}_X(1)$. \blacksquare

Lemma 2.1.3. *Let $X \hookrightarrow \mathbb{P}(V)$ be a T -linearized projective variety. There exists only finitely subtori $T' \subseteq T$ occurring as $(\text{stab } x)^0$ for $x \in X_T^{ss}$.*

Proof. Let $X \hookrightarrow \mathbb{P}(V)$ via some high tensor power of the T -linearization. A rank r subtorus $T' \subseteq T$ stabilizes $x \in X$ if and only if each element $\lambda \in \Lambda_*(T') \subseteq \Lambda_*(T)$ is constant when paired

with the elements of $\Xi(x) \subseteq \Lambda^*(T)$. Therefore, $(\text{stab } x)^0$ is the subtorus of T corresponding to the subgroup of 1-parameter subgroups that are constant on $\Xi(x)$. The set of states of $x \in X$ is a finite set since V is a finite-dimensional representation, and therefore the tori that appear as $(\text{stab } x)^0$ form a finite collection. \blacksquare

2.2. Constructing Y . The trick will be to define Y to be $Y := \mathbb{P}(V) \times (G/B)^{\text{rank } G}$, the iterated product of $\mathbb{P}(V)$ with a flag variety. The only delicate point is the choice of a suitable linearization. Let $T \subseteq B \subseteq G$ be a choice of Borel subgroup containing a split maximal torus T .

Lemma 2.2.1. *Let $\chi \in \Lambda^*(T)$ be in the positive Weyl chamber. The G -equivariant line bundle $L(\chi) := G \times_{B, \chi} \mathbb{A}^1$ on G/B is ample and hence a G -linearization of G/B . For any nontrivial subtorus $T' \subseteq T$ that fixes a point $gB \in G/B$, the induced T' -action on the fibre $L(\chi)|_{gB}$ is by weight $(w \cdot \chi)|_{T'}$, where $w \in W$ corresponds to the Bruhat cell containing gB .*

Proof. $L(\chi)$ is ample on G/B by the Borel-Weil-Bott theorem. By the Bruhat decomposition, $G/B = \coprod_{w \in W} U_w B$, for $U \subseteq B$ the maximal unipotent subgroup. Moreover, denoting by $\dot{w} \in N_T$ a lift of the element $w \in W$, there is an isomorphism $\phi : U_w \times B \rightarrow U\dot{w}B \subseteq G$ sending $(u, b) \mapsto u\dot{w}b$, where $U_w := U \cap (U^-)^w$ is the intersection of U with the w -conjugate of the opposite unipotent subgroup (cf. [Bo, Thm. 14.12]). Note that U_w is normalized by T because both U and $(U^-)^w$ are; the latter being so because for any $t \in T$, $\dot{w} \in N_T$, and $u \in U^-$,

$$t(\dot{w}^{-1}u\dot{w})t^{-1} = \dot{w}^{-1}(t^{\dot{w}}u(t^{\dot{w}})^{-1})\dot{w}.$$

Assume $gB = u\dot{w}B$ for some $u \in U_w$ and $\dot{w} \in N_T$. Let $t \in T'$ be an element fixing gB and observe

$$tu\dot{w} = tut^{-1}\dot{w}t^{\dot{w}-1},$$

with $tut^{-1} \in U_w$ and $t^{\dot{w}-1} = \dot{w}^{-1}t\dot{w} \in T \subseteq B$. Since t fixes gB , there exists some $b \in B$ such that $u\dot{w}b = tut^{-1}\dot{w}t^{\dot{w}-1}$, and since ϕ is an isomorphism, $u = tut^{-1}$ and $b = t^{\dot{w}-1}$. Therefore $tu\dot{w} = u\dot{w}t^{\dot{w}-1}$, and so t acts on the fibre by multiplication by $\chi(t^{\dot{w}-1}) = (w \cdot \chi)(t)$. \blacksquare

The following lemma will be the inductive step in the argument showing that for a G -linearized variety Z (e.g. $Z = \mathbb{P}(V)$), the variety $Y := Z \times (G/B)^{\text{rank } G}$ admits a linearization such that $Y_T^{ss} = Y_T^s$ and all points $y \in Y$ projecting to a stable $z \in Z$ are stable in Y .

Lemma 2.2.2. *Let \mathcal{L} be a G -linearization of a projective variety Z . If the subtori of a split maximal torus $T \subseteq G$ occurring as $(\text{stab}_T z)^0$ for $z \in Z_T^{ss}$ are at most rank $r > 0$, then there is a character $\chi \in \Lambda^*(T)$ such that the G -equivariant line bundle $L(\chi) := G \times_{B, \chi} \mathbb{A}^1$ is very ample on G/B , and for some $N \gg 0$, the induced G -linearized action on the line bundle $\mathcal{L}^{\otimes N} \boxtimes L(\chi)$ on $Z \times G/B$ has the properties:*

- (1) *The subtori of T of the form $(\text{stab}_T p)^0$ for $p \in (Z \times G/B)_T^{ss}$ are at most rank $r - 1$;*
- (2) *A point $p \in Z \times G/B$ is G - or T -stable (resp. G - or T -unstable) whenever $\pi(p)$ is so, where $\pi : Z \times G/B \rightarrow Z$ is projection onto the first factor.*

Proof. Let T_1, \dots, T_n denote the positive-dimensional subtori of T occurring as $(\text{stab}_T z)^0$ for $z \in Z_T^{ss}$. Let $H_i \subset \Lambda^*(T)$ be dual to the subgroup $\Lambda_*(T_i) \subseteq \Lambda_*(T)$. Since each T_i is positive dimensional, all of the H_i are proper subgroups of $\Lambda^*(T)$. Choose χ in the interior of the positive Weyl chamber of $\Lambda^*(T)$ avoiding the W -orbit of any H_1, \dots, H_n , and large enough so that $L(\chi)$ is very ample on G/B .

The state $\Xi(p)$ of a point $p = (z, gB) \in Z \times G/B$ with respect to the linearization $\mathcal{L}^{\otimes N} \boxtimes L(\chi)$ consists of weights of the form $N \cdot \chi_z + \chi_{gB}$, with $\chi_z \in \Xi(z)$ and $\chi_{gB} \in \Xi(gB)$. By choosing N

large enough, the Hilbert-Mumford criterion shows that the induced linearized action on $\mathcal{L}^{\otimes N} \boxtimes L(\chi)$ satisfies:

- ◊ $p \in Z \times G/B$ is stable if $\pi(p)$ is stable;
- ◊ $p \in Z \times G/B$ is unstable if $\pi(p)$ is unstable.

This shows (2), and moreover that any strictly semi-stable point $p \in Z \times G/B$ sits above a strictly semi-stable point $\pi(p) \in Z$.

With the prescribed G -linearized action on $\mathcal{L}^{\otimes N} \boxtimes L(\chi)$, let $T' := (\text{stab}_T p)^0$ for some point $p := (z, gB) \in (Z \times G/B)_T^{\text{ss}}$. As we just noted, z is strictly semi-stable in Z and so T' is contained in T_i for some i . Recall that χ was chosen so that $(w \cdot \chi)|_{T_i} \neq 0 \in \Lambda^*(T_i)$ for any $w \in W$. Since p is strictly semi-stable, by Lemma 2.1.2 the weight of the induced linearization of T' at (z, gB) will be 0. The weight of the action of T' on $L(\chi)|_{gB}$ is $w \cdot \chi$ for some $w \in W$, by Lemma 2.2.2. Therefore, the weight of the linearization of T' at (z, gB) is $0 + \chi|_{T'}$; hence $T' \subseteq \ker(\chi|_{T_i})$, and therefore the rank of T' is at most $r - 1$. ■

Proposition 2.2.3. *If Z is a G -linearized projective variety then $Y := Z \times (G/B)^r$ for $r := \text{rank } G$ admits a G -linearization for which:*

- (i) $Y^{\text{ss}} = Y^s$; and
- (ii) $Z^s \times (G/B)^r \subseteq Y^{\text{ss}} \subseteq Z^{\text{ss}} \times (G/B)^r$,

for both T - and G - (semi-)stability.

Proof. We prove the result for T -stability, and then as $Y_G^{\text{ss}} = \bigcap_{g \in G} Y_T^{\text{ss}}$ and $Y_G^s = \bigcap_{g \in G} Y_T^s$, the results for G -stability will follow (cf. Theorem 1.1.4). Recursively applying Lemma 2.2.2, we obtain a G -linearization of $Z \times (G/B)^{\times \text{rank } G}$ for which no T -strictly semi-stable points have positive dimensional T -stabilizers. By Lemma 2.1.1, there are no T -strictly semi-stable points, proving (i). Lemma 2.2.2(2) implies (ii). ■

2.3. $\sqrt{c_{\text{top}}}$ on singular X .

Lemma 2.3.1. *If $\pi : Y \rightarrow X$ be a surjective proper morphism between varieties, then the push-forward map $\pi_* : A_*(Y)_{\mathbb{Q}} \rightarrow A_*(X)_{\mathbb{Q}}$ is a surjective map between Chow groups with rational coefficients.*

Proof. Take a subvariety $Z \hookrightarrow X$. Let ξ_Z be the generic point. Then let $Z' \hookrightarrow Y$ be the scheme-theoretic closure of any closed point in the fibre $Y \times_X \kappa(\xi_Z)$. The scheme Z' sits generically finitely over Z , and the class $\frac{1}{[K(Z') : K(Z)]} [Z']$ pushes forward to $[Z]$. Therefore π_* is surjective. ■

We have developed enough theory to prove the main result of this section:

Proposition 2.3.2. *Let $j : X \hookrightarrow Z$ be a G -equivariant inclusion of varieties with compatible G -linearizations such that $X_T^{\text{ss}} = X_T^s$. For $\pi : Y := Z \times (G/B)^r \rightarrow Z$ with the G -linearization of Proposition 2.2.3, and any $\alpha \in A_*(X//G)_{\mathbb{Q}}$ with lift $\tilde{\alpha} \in A_*(X//T)_{\mathbb{Q}}$, there exists a class $\beta \in A_*(Y//G)_{\mathbb{Q}}$ with lift $\tilde{\beta} \in A_*(Y//T)_{\mathbb{Q}}$ so that*

- (1) $\int_{X//G} \alpha = \int_{Y//G} \beta$;
- (2) $\int_{X//T} \sqrt{c_{\text{top}}} \frown \tilde{\alpha} = \int_{Y//T} \sqrt{c_{\text{top}}} \frown \tilde{\beta}$.

Proof. Let $X'_H := X_H^{\text{ss}} \times_X Y_H^{\text{ss}}$, where $H = T$ or $H = G$. Proposition 2.2.3 is enough to guarantee that X'_H embeds T -equivariantly as a closed subvariety of Y_H^{ss} and T -equivariantly and surjectively

onto X_H^{ss} . The following commutative diagram of proper Deligne-Mumford stacks,

$$\begin{array}{ccc} [X'_H/H] & \xrightarrow{j} & [Y_H^{ss}/H] \\ \pi \downarrow & & \downarrow \\ [X_H^{ss}/H] & \longrightarrow & \text{Spec } k, \end{array}$$

induces an analogous diagram between the coarse moduli spaces. By Lemma 2.3.1, $\pi_*\alpha' = \alpha$ for some $\alpha' \in A_*([X'_G/G])_{\mathbb{Q}}$. Because Z'_H is a fibre product, $\pi_*(\tilde{\alpha}')$ is a lift of α for any lift $\tilde{\alpha}'$ of α' . Let $\beta := j_*(\alpha') \in A_*(Y//G)_{\mathbb{Q}}$ and $\tilde{\beta} := j_*(\tilde{\alpha}') \in A_*(Y//T)_{\mathbb{Q}}$. By the commutativity of the diagram, the degrees of the classes α and β agree, proving (1). The equality in (2) follows similarly. \blacksquare

Finally, we prove the result advertised in the introduction to this section.

Proof of Corollary 2.0.1. Embed the singular X into the smooth variety $\mathbb{P}(V)$ via some high tensor power of the given G -linearization. Construct the smooth G -linearized Y sitting over $\mathbb{P}(V)$. By Proposition 2.3.2, any two lifts $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ of a class $\alpha \in A_*(X//G)_{\mathbb{Q}}$ have analogues $\tilde{\beta}_0$ and $\tilde{\beta}_1$ both lifting a class $\beta \in A_*(Y//G)_{\mathbb{Q}}$ and satisfying

$$\int_{X//T} \sqrt{c_{\text{top}}} \frown \tilde{\alpha}_i = \int_{Y//T} \sqrt{c_{\text{top}}} \frown \tilde{\beta}_i,$$

for $i = 0, 1$. The result immediately follows from the smooth case (cf. Corollary 1.5.3). \blacksquare

3. INDEPENDENCE OF GIT INTEGRAL RATIO

The goal of this section is to prove Theorem 0.0.4: for a G -linearized projective variety X over a field k with no strictly T -semi-stable points, the ratio $r_{G,T}^{X,\alpha}$, (defined below in §3.1) is an invariant of the group G . We do this in stages, first showing that it is independent of the rational equivalence class $\alpha \in A_0(X//G)_{\mathbb{Q}}$ and the choice of maximal torus, before showing the independence on the variety X .

3.1. Independence on Chow class. By Corollary 2.0.1, the following ratio is well-defined:

Definition 3.1.1. Assume X is a G -linearized projective variety, $T \subseteq G$ is a split maximal torus for which $X_T^{ss} = X_T^s$, and $\alpha \in A_0(X//G)$ is a 0-cycle satisfying $\int_{X//G} \alpha \neq 0$. We define the *GIT integral ratio* to be

$$r_{G,T}^{X,\alpha} := \frac{\int_{X//T} c_{\text{top}} \frown \tilde{\alpha}}{\int_{X//G} \alpha},$$

where $\tilde{\alpha}$ is some lift of the class α .

Lemma 3.1.2. The GIT integral ratio $r_{G,T}^X := r_{G,T}^{X,\alpha}$ is independent of the choice of Chow class $\alpha \in A_0(X//G)_{\mathbb{Q}}$.

Proof. The definition of $r_{G,T}^{X,\alpha}$ is independent of the algebraic equivalence class of α , since numerical equivalence is coarser than algebraic equivalence. Let $B_*(-)$ denote the quotient of the Chow group $A_*(-)$ by the relation of algebraic equivalence (cf. [Fu, §10.3]). Since $[X_G^{ss}/G]$ is a Deligne-Mumford stack $B_0([X_G^{ss}/G])_{\mathbb{Q}} = B_0(X//G)_{\mathbb{Q}}$. All connected projective schemes are algebraically connected, i.e. there is a connected chain of (possibly singular) curves connecting any two closed points. Therefore, $B_0(X//G)_{\mathbb{Q}} = \mathbb{Q}$, and the result follows since $r_{G,T}^{X,\alpha}$ is invariant under the scaling of α . \blacksquare

3.2. Independence on split maximal torus.

Lemma 3.2.1. *The GIT integral ratio $r_G^X := r_{G,T}^X$ does not depend on the choice of split maximal torus T .*

Proof. Fix two split maximal tori $T, T' \subseteq G$. All split maximal tori are conjugate, and so T' is of the form $T' = gTg^{-1}$ for some $g \in G$. By assumption, G acts linearly on the projective variety X . Consider the map $\phi : T \rightarrow T'$ given by $t \mapsto gtg^{-1}$, and the map $\Phi : X \rightarrow X$ given by $x \mapsto x \cdot g^{-1}$. The pair of maps (ϕ, Φ) show that the linearized actions $\sigma : X \times T \rightarrow X$ and $\sigma' : X \times T' \rightarrow X$ are isomorphic: $\Phi(x) \cdot \phi(t) = \Phi(x \cdot t)$. By the Hilbert-Mumford numerical criterion, $X_{T'}^{ss} = \Phi(X_T^{ss})$, and furthermore there is an induced isomorphism $\bar{\Phi} : [X_{T'}^{ss}/T'] \cong [X_T^{ss}/T]$. Since the following square is commutative,

$$\begin{array}{ccc} X_G^{ss} & \xrightarrow{\bar{\Phi}} & X_G^{ss} \\ i_T \downarrow & & \downarrow i_{T'} \\ X_T^{ss} & \xrightarrow{\bar{\Phi}} & X_{T'}^{ss}, \end{array}$$

the push-forward $\bar{\Phi}_* \tilde{\alpha}$ is a lift of $\bar{\Phi}_* \alpha$ for any lift $\tilde{\alpha}$ of $\alpha \in A_0(X//G)_{\mathbb{Q}}$. We use the pairs $(\alpha, \tilde{\alpha})$ and $(\Phi_* \alpha, \Phi_* \tilde{\alpha})$ to compute the GIT integral ratios. Since $\bar{\Phi}$ is an isomorphism, for any class β , the class $\bar{\Phi}_* \beta$ has the same degree. Therefore, the ratios $r_{G,T}^{X,\alpha}$ and $r_{G,T'}^{X,\Phi_* \alpha}$ are equal. \blacksquare

3.3. Independence on linearized variety.

Lemma 3.3.1. *The GIT integral ratio $r_G := r_G^X$ does not depend on the choice of G -linearized variety X .*

Proof. Let X_i for $i = 1, 2$ be two G -linearized projective varieties for which $(X_i)_T^{ss} = (X_i)_T^s$. Some high tensor powers of these linearizations define G -equivariant embeddings $j_i : X_i \hookrightarrow \mathbb{P}(V_i)$ for G -representations V_i , $i = 1, 2$. For $i = 1, 2$ there are embeddings $X_i \hookrightarrow \mathbb{P}(V_1 \oplus V_2)$, defined from the embeddings j_i by setting the extraneous coordinates to 0. By Proposition 2.3.2, for any classes $\alpha_i \in A_0(X_i//G)_{\mathbb{Q}}$, there are classes $\beta_i \in A_0(Y//G)_{\mathbb{Q}}$, for a smooth G -linearized Y over $\mathbb{P}(V)$ satisfying $Y_T^{ss} = Y_T^s \neq \emptyset$, such that $r_G^{Y,\beta_i} = r_G^{X_i,\alpha_i}$. By Lemma 3.1.2, $r_G^{Y,\beta_1} = r_G^{Y,\beta_2}$ is an invariant of the G -linearized space Y , and therefore $r_G^{X_1,\alpha_1} = r_G^{X_2,\alpha_2}$. \blacksquare

Proof of Theorem 0.0.4. The above lemmas combine to prove r_G is an invariant of the group G for any reductive group G with a split maximal torus T over a field k . \blacksquare

4. FUNCTORIAL PROPERTIES OF THE GIT INTEGRATION RATIO

In this section, we prove that the GIT integration ratio behaves well with respect to the group operations of direct product and central extension.

4.1. Field extensions and direct products.

Lemma 4.1.1. *If G is a reductive group over k and G_K its base change by a field extension K/k , then the GIT integration ratios for G and G_K are equal:*

$$r_G = r_{G_K}.$$

Proof. If X is a projective variety over k with a G -linearization, then there is an induced G_K -linearized action on X_K . By [FKM, Prop. 1.14], $X_G^{ss} \times K = (X_K)_{G_K}^{ss}$ (and similarly for T). It follows that $(X//G)_K \cong X_K//G_K$ and $(X//T)_K \cong X_K//T_K$. The result can be deduced from the

facts that the degree of a Chow class is invariant under field extension [Fu, Ex. 6.2.9] and that c_{top} pulls-back to c_{top}^K by the natural morphism $BT_K \rightarrow BT$. \blacksquare

Lemma 4.1.2. *If G_1, G_2 are two reductive groups over a field k , then $r_{G_1 \times G_2} = r_{G_1} \cdot r_{G_2}$.*

Proof. For each $i = 1, 2$, choose a projective X_i on which G_i acts linearly, and let $T_i \subseteq G_i$ denote split maximal tori. Clearly $G_1 \times G_2$ acts on $X_1 \times X_2$ linearly, and the stability loci are just the products of the corresponding loci from the factors. Let $\alpha_i \in A_0^{G_i}((X_i)_{G_i}^{ss})$, and consider $\alpha := \alpha_1 \times \alpha_2 \in A_0^{G_1 \times G_2}((X_1)_{G_1}^{ss} \times (X_2)_{G_2}^{ss})$. Also, take $\tilde{\alpha} := \tilde{\alpha}_1 \times \tilde{\alpha}_2$ to where each α_i is be the lift of α_i to the T_i -semi-stable locus of X_i . We may calculate the GIT integration ratio for $G_1 \times G_2$ using these classes, since the ratio is independent of such choices (cf. Theorem 0.0.4). The degree of a product of two classes is the product of the degrees, and so the result follows since the isomorphism $[\text{Spec } k/T] \cong [\text{Spec } k/T_1] \times_k [\text{Spec } k/T_2]$ identifies $c_{\text{top}}(T_1) \times c_{\text{top}}(T_2) = c_{\text{top}}(T)$. \blacksquare

4.2. Central extensions. In this section we prove that r_G is invariant under central extensions, completing the proof of Theorem 0.0.5. Throughout the course of the discussion, we will be working with several types of quotients. If G acts on the right on a variety X , when the respective quotients exist, they will be denoted as follows: the stack-theoretic quotient by $[X/G]$, the GIT quotient by $X//G$, and the uniform categorical quotient by X/G .

Lemma 4.2.1. *Let \mathcal{L} be a G -linearization of a variety X for which the GIT quotient $\pi : X_G^{ss} \rightarrow X//G$ is nonempty. There exists some $n > 0$ such that $\mathcal{L}^{\otimes n}$ descends to a line bundle $\widehat{\mathcal{L}}$ (i.e. $\pi^* \widehat{\mathcal{L}} = \mathcal{L}^{\otimes n}|_{X_G^{ss}}$), and moreover, $\widehat{\mathcal{L}}$ is the uniform categorical quotient of the induced linearization $G \times \mathcal{L}^{\otimes n} \rightarrow \mathcal{L}^{\otimes n}$.*

Proof. From [FKM, Thm. 1.10], we see that $X_G^{ss} \rightarrow X//G$ is a uniform categorical quotient and that some power $\mathcal{L}^{\otimes n}$ descends to $\widehat{\mathcal{L}}$. Since $\mathcal{L} \rightarrow X//G$ is flat, the base change morphism $\mathcal{L}^{\otimes n} \rightarrow \widehat{\mathcal{L}}$ is also a uniform categorical quotient. \blacksquare

Lemma 4.2.2. *Let $1 \rightarrow S \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ be a central extension of reductive groups, X a scheme, $\pi_{\tilde{G}} : X \rightarrow X/\tilde{G}$ and $\pi_S : X \rightarrow X/S$ uniform categorical quotients by \tilde{G} and S respectively, and $\pi_G : X/S \rightarrow X/\tilde{G}$ the induced morphism. If X/\tilde{G} is covered by finitely many affine open subschemes U for which both $\pi_{\tilde{G}}^{-1}(U)$ and $\pi_G^{-1}(U)$ are affine, then π_G is a uniform categorical quotient by the induced action of G on X/S .*

Proof. Consider the action $\sigma : X \times \tilde{G} \rightarrow X$. Compose with π_S to obtain $X \times \tilde{G} \rightarrow X/S$. This is an $S \times S$ -invariant morphism and hence descends to an action $X/S \times G \rightarrow X/S$. All that remains to be shown is that $\pi_G : X/S \rightarrow X/\tilde{G}$ is a uniform categorical quotient for this action.

By [FKM, Rem. 0.2(5)], it suffices to show for each U described in the lemma statement that the restriction $\pi_G : \pi_G^{-1}(U) \rightarrow U$ is a uniform categorical quotient. This boils down to the easy fact that for an affine ring R on which \tilde{G} acts, the rings of invariants satisfy $R^{\tilde{G}} = (R^S)^G$. \blacksquare

Lemma 4.2.3. *Let $1 \rightarrow S \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ be a central extension of reductive groups. Let \tilde{G} act linearly on a variety X , and $\pi : X_S^{ss} \rightarrow X//S$ denote the induced GIT quotient. Then there is an induced G -linearized action on $X//S$ and the semi-stable loci satisfy*

$$(4.2.4) \quad X_G^{ss} = \pi^{-1}((X//S)_G^{ss}).$$

Moreover, this yields a canonical isomorphism between GIT quotients $(X//S)//G \cong X//\tilde{G}$.

Proof. Choose $n \in \mathbb{N}$ as in Lemma 4.2.1 so that $\mathcal{L}^{\otimes n}$ descends to the ample line bundle $\widehat{\mathcal{L}}$ on $X//S$ that is a uniform categorical quotient of $\mathcal{L}^{\otimes n}$ by S . Lemma 4.2.2 results in compatible G -actions on $X//S$ and $\mathcal{L}^{\otimes n}$, i.e. a G -linearized action on $X//S$. Therefore, we can take the GIT quotient $(X//S)//G$, which by Lemma 4.2.2 is a uniform categorical \tilde{G} -quotient of $\pi^{-1}((X//S)_G^{ss})$. Since $\widehat{\mathcal{L}}$ pulls-back to a tensor power of \mathcal{L} , and G -equivariant sections of $\widehat{\mathcal{L}}^{\otimes m}$ pull-back to \tilde{G} -equivariant sections of $\mathcal{L}^{\otimes mn}$, we have $\pi^{-1}(X//S)_G^{ss} \subseteq X_G^{ss}$. Conversely, if $\sigma \in \Gamma(X, \mathcal{L}^{\otimes n})^{\tilde{G}}$ is a \tilde{G} -equivariant section, then due to its S -equivariance, it descends to a G -equivariant section $\bar{\sigma} \in \Gamma(X//S, \widehat{\mathcal{L}})^G$ since $\widehat{\mathcal{L}}$ is a quotient of $\mathcal{L}^{\otimes n}$ by S . Therefore, the inclusion is full: $\pi^{-1}((X//S)_G^{ss}) = X_G^{ss}$. Combining this with Lemma 4.2.2, we obtain the equalities

$$X//\tilde{G} = (X_G^{ss}/S)/G = (X//S)_G^{ss}/G = (X//S)//G.$$

■

Lemma 4.2.5. *Let $1 \rightarrow S \rightarrow \tilde{G} \twoheadrightarrow G \rightarrow 1$ is a central extension of reductive groups. If \tilde{G} acts linearly on X so that $X_G^{ss} = X_{\tilde{G}}^{ss} \neq \emptyset$, then there is a canonical morphism of stacks $\phi : [X_G^{ss}/\tilde{G}] \rightarrow [(X_G^{ss}/S)/G]$. Moreover, the morphism ϕ is proper and makes the following diagram commute:*

$$\begin{array}{ccc} [X_G^{ss}/\tilde{G}] & \xrightarrow{\phi} & [(X_G^{ss}/S)/G] \\ \downarrow & & \downarrow \\ [\mathrm{Spec} k/\tilde{G}] & \longrightarrow & [\mathrm{Spec} k/G]. \end{array}$$

Proof. We describe the functor ϕ on objects, omitting the description of the functor on morphisms. Let $\underline{E} := (B \xleftarrow{\pi} E \xrightarrow{f} X_G^{ss})$ be an object of $[X_G^{ss}/\tilde{G}]$, i.e. $\pi : E \rightarrow B$ is a \tilde{G} -torsor in the étale topology, and $f : E \rightarrow X_G^{ss}$ is a \tilde{G} -equivariant morphism. Consider the scheme $E \times_{\tilde{G}} G$, which exists by the descent of affine morphisms in the étale topology. Furthermore, by the descent of morphisms, it is clear that this is the uniform categorical quotient E/S , and hence maps to X_G^{ss}/S .

This results in an object $\underline{E} \times_{\tilde{G}} G := (B \xleftarrow{\pi} E \times_{\tilde{G}} G \xrightarrow{f} X_G^{ss}/S)$ of the stack $[(X_G^{ss}/S)/G]$. The construction $\underline{E} \mapsto \underline{E} \times_{\tilde{G}} G$ is clearly functorial. Therefore ϕ is a functor between categories fibred in groupoids and hence a morphism of stacks. Moreover, since the morphism of stacks $[\mathrm{Spec} k/\tilde{G}] \rightarrow [\mathrm{Spec} k/G]$ is defined on B -points by $(E \xrightarrow{\pi} B) \mapsto (E \times_{\tilde{G}} G \xrightarrow{\pi} B)$, the above square commutes.

To see that ϕ is proper, consider the morphisms to the coarse moduli spaces $f : [X_G^{ss}/\tilde{G}] \rightarrow X//\tilde{G}$ and $g : [(X_G^{ss}/S)/G] \rightarrow X//G$. By Lemma 4.2.3, these morphisms are well-defined and $f = g \circ \phi$. Being coarse moduli morphisms, in particular f is proper and g is separated, so consequently ϕ is proper. ■

Proposition 4.2.6. *If $\tilde{G} \twoheadrightarrow G$ is a central extension of reductive groups, then $r_{\tilde{G}} = r_G$.*

Proof. The case of a finite central extension is trivial, as we may use the same G -variety X on which to calculate both ratios r_G and $r_{\tilde{G}}$. This allows us to then reduce to the case where the kernel S of the central extension is connected. Since S centralizes the maximal torus $\tilde{T} \subseteq \tilde{G}$, it must be contained within \tilde{T} ; hence there is also an analogous exact sequence involving the maximal tori, $1 \rightarrow S \rightarrow \tilde{T} \rightarrow T \rightarrow 1$. Also, notice that the class $c_{\mathrm{top}}(G) \in A^*(BT)$ pulls-back via $B\tilde{T} \rightarrow BT$ to $c_{\mathrm{top}}(\tilde{G}) \in A^*(B\tilde{T})$, since the map $\tilde{G} \twoheadrightarrow G$ induces an isomorphism of root systems. Let X be a projective \tilde{G} -linearized variety such that $X_{\tilde{T}}^{ss} = X_T^{ss}$ and $X_G^{ss} \neq \emptyset$. By Lemma 4.2.3, the projective

scheme $X//S$ has an induced G -linearization for which it is easy to see $(X//S)_T^{ss} = (X//S)_T^s$ and $(X//S)_G^{ss} \neq \emptyset$. We use X to compute $r_{\tilde{G}}$ and $X//S$ to compute r_G .

There is a commutative diagram:

$$\begin{array}{ccc}
 [X_{\tilde{T}}^{ss}/\tilde{T}] & \xrightarrow{\phi} & [(X//S)_T^{ss}/T] \\
 \uparrow i & & \uparrow i \\
 [X_{\tilde{G}}^{ss}/\tilde{T}] & \xrightarrow{\phi} & [(X//S)_G^{ss}/T] \\
 \downarrow \pi & & \downarrow \pi \\
 [X_{\tilde{G}}^{ss}/\tilde{G}] & \xrightarrow{\phi} & [(X//S)_G^{ss}/G].
 \end{array}$$

Since Lemma 4.2.3 applies equally well to quasi-projective varieties X , combining with Lemma 4.2.5, we see that each morphism ϕ induces an isomorphism ϕ_* on rational Chow groups. Moreover, the commutative diagram of Lemma 4.2.5 implies that $\phi_*(c_{\text{top}} \frown \tilde{\alpha}) = c_{\text{top}} \frown \phi_*\tilde{\alpha}$. The theorem follows once we check that $\phi_*(\tilde{\alpha})$ is a lift of the class $\phi_*\alpha$, for then the equality of the ratios

$$\frac{\int_{X//\tilde{T}} c_{\text{top}} \frown \tilde{\alpha}}{\int_{X//\tilde{G}} \alpha} = \frac{\int_{(X//S)//T} c_{\text{top}} \frown \widetilde{\phi_*\alpha}}{\int_{(X//S)//G} \phi_*\alpha}$$

follows immediately from the functoriality of Chow groups under proper push-forwards. The fact that $\phi_*\tilde{\alpha}$ is a lift of $\phi_*\alpha$ will follow from the equality

$$\phi_*(\pi^*\alpha) = \pi^*(\phi_*\alpha),$$

which follows from the standard push-pull argument once we prove that the lower square is a fibre square of DM stacks.

Since the lower square of the diagram commutes, there is a functor from $[X_{\tilde{G}}^{ss}/\tilde{T}]$ to the fibre product. All that remains is to construct an inverse functor that would demonstrate an equivalence of categories. We do so, explicitly describing the functor on objects, but again omitting the details of the definition on morphisms. Given a \tilde{G} -torsor $(B \leftarrow \tilde{E} \rightarrow X_{\tilde{G}}^{ss})$, a T -torsor $(B \leftarrow E \rightarrow (X//S)_G^{ss})$, and an isomorphism of G -torsors $\tilde{E}/S \cong E \times_T G$, we must construct a \tilde{T} -torsor and a \tilde{T} -equivariant morphism to $X_{\tilde{G}}^{ss}$. This can be accomplished by taking $E \times_{\tilde{E}/S} \tilde{E}$, where one of the structure morphisms is the composition $E \rightarrow E \times \{e\} \rightarrow E \times_T G \cong \tilde{E}/S$ and the other is the quotient $\tilde{E} \rightarrow \tilde{E}/S$. The \tilde{T} equivariant morphism $E \times_{\tilde{E}/S} \tilde{E} \rightarrow X_{\tilde{G}}^{ss}$ is the composition of the projection onto \tilde{E} with the \tilde{G} -equivariant morphism to $X_{\tilde{G}}^{ss}$. It is routine to verify that this is indeed the inverse functor. \blacksquare

These results combine to prove Theorem 0.0.5.

Proof of 0.0.5. Immediate from the results of this section. \blacksquare

5. $r_G = |W|$ FOR GROUPS OF TYPE A_n

We compute the GIT integration ratio r_G for $G = PGL(n)$, and use this to prove Corollary 0.0.6.

Proposition 5.0.1. *The GIT integration ratio for $G = PGL(n)$ over any field is $r_G = |W| = n!$.*

Proof. Let $SL(n)$ act on \mathbf{M}_n , the vector space of $n \times n$ matrices with k -valued entries, via left multiplication of matrices. This induces a dual representation of $SL(n)$ on \mathbf{M}_n^* and hence actions of $SL(n)$ and $PGL(n)$ on $\mathbb{P}(\mathbf{M}_n)$, the projective space of lines in \mathbf{M}_n^* . Choose the $PGL(n)$ -linearization on $\mathcal{O}_{\mathbb{P}(\mathbf{M}_n)}(n)$ induced from these representations of $SL(n)$. Let $T \subseteq PGL(n)$ and $\tilde{T} \subseteq SL(n)$ denote the diagonal maximal tori. In this case, the $PGL(n)$ -stability loci (resp. T -stability loci) are equal to the analogous $SL(n)$ -stability loci (resp. \tilde{T} -stability loci), which we now describe.

A basis of \mathbf{M}_n is given by the matrices e_{ij} , each defined by its unique nonzero entry of 1 in the (i, j) th position. Moreover, e_{ij} is a weight vector of weight $\chi_i \in \Lambda^*(\tilde{T})$, where χ_i is defined by the rule

$$\begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{bmatrix} \mapsto t_i.$$

Notice that $\chi_1, \dots, \chi_{n-1}$ is a basis of the character group $\Lambda^*(\tilde{T})$ and $\chi_n = -\sum_{i=1}^{n-1} \chi_i$, so that the characters χ_1, \dots, χ_n form the vertices of a simplex centered at the origin in $\Lambda^*(\tilde{T})_{\mathbb{Q}}$. From the Hilbert-Mumford criterion (cf. Theorem 1.1.4), one quickly concludes that the \tilde{T} -unstable locus in $\mathbb{P}(\mathbf{M}_n)$ is the set of all points $x \in \mathbb{P}(\mathbf{M}_n)$ such that the matrix $e_{ij}(x)$ has a row with all entries 0; all other points are \tilde{T} -stable. Thus, there are no strictly semi-stable points for the \tilde{T} -action, and hence neither are there any for the $SL(n)$ -action. The $SL(n)$ -stable locus then comprises the set of $x \in \mathbb{P}(\mathbf{M}_n)$ such that the matrix $e_{ij}(x)$ is of full-rank; this comprises a dense $PGL(n)$ orbit. The stabilizer of this orbit is trivial, so $[\mathbb{P}(\mathbf{M}_n)^{ss}_{PGL(n)}/PGL(n)] \cong \mathbb{P}(\mathbf{M}_n)//PGL(n) \cong \text{Spec } k$. The T -quotient is clearly $[\mathbb{P}(\mathbf{M}_n)^{ss}_T/T] \cong \mathbb{P}(\mathbf{M}_n)//T \cong (\mathbb{P}^{n-1})^n$.

The rational Chow ring of the T -quotient is

$$A^*((\mathbb{P}(\mathbf{M}_n)//T)_{\mathbb{Q}}) \cong \mathbb{Q}[t_1, \dots, t_n]/(t_1^n, \dots, t_n^n).$$

In this ring, the class of a point is clearly $\prod_{i=1}^n t_i^{n-1}$. The class $c_{\text{top}} \in A^*(BT)$ is the product of all the roots, which are of the form $\alpha_{ij} := \chi_i - \chi_j \in \text{Sym}^* \Lambda^*(T) \cong A^*(BT)$, for $1 \leq i \neq j \leq n$. One can easily check that the pull-back of $\chi_i - \chi_j$ to $A^*(\mathbb{P}(\mathbf{M}_n)//T)_{\mathbb{Q}}$ is $t_i - t_j$, and therefore $c_{\text{top}} = \prod_{i \neq j} (t_i - t_j)$. Let $\alpha \in A_0(\mathbb{P}(\mathbf{M}_n)//G)_{\mathbb{Q}} \cong \mathbb{Q}$ denote the fundamental class; i.e. $\int_{\mathbb{P}(\mathbf{M}_n)//PGL(n)} \alpha = 1$. Therefore, the GIT integral ratio r_G is just $\int_{\mathbb{P}(\mathbf{M}_n)//T} c_{\text{top}}$, which equals the coefficient of the monomial $\prod_{i=1}^n t_i^{n-1}$ in the expansion of $\prod_{i \neq j} (t_i - t_j)$, since all other monomials of degree $n^2 - n$ are 0 in the ring $\mathbb{Q}[t_1, \dots, t_n]/(t_1^n, \dots, t_n^n)$.

Notice that $\prod_{i \neq j} (t_i - t_j) = (-1)^{n(n-1)/2} (\det M_V)^2$, where $\det M_V$ is the determinant of the Vandermonde matrix

$$M_V := \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix}.$$

By definition, $\det M_V = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n t_i^{\sigma(i)-1}$. In the ring $A^*((\mathbb{P}^{n-1})^n)_{\mathbb{Q}}$, we compute the products of monomials of the form $m_{\sigma} := \prod_{i=1}^n t_i^{\sigma(i)-1}$ for $\sigma \in S_n$:

$$m_{\sigma} \cdot m_{\sigma'} = \begin{cases} \prod_{i=1}^n t_i^{n-1} & : \sigma(j) + \sigma'(j) = n + 1; \forall 1 \leq j \leq n; \\ 0 & : \text{otherwise.} \end{cases}$$

If $w_0 := (1\ n)(2\ n-1) \cdots (\lceil n/2 \rceil\ \lceil (n+1)/2 \rceil) \in S_n$ denote the longest element of the Weyl group W , then for each $\sigma \in S_n$, the permutation σ' defined as the composition $\sigma' := w_0 \circ \sigma$ is the unique element of S_n for which $m_\sigma \cdot m_{\sigma'} \neq 0$. For such pairs (σ, σ') , the product of the signs satisfies $\text{sgn}(\sigma) \cdot \text{sgn}(\sigma') = \text{sgn}(w_0) = (-1)^{(n^2-n)/2}$. Therefore,

$$\begin{aligned} c_{\text{top}} &= (-1)^{(n^2-n)/2} \cdot \sum_{\sigma \in S_n} (-1)^{(n^2-n)/2} \prod_{i=1}^n t_i^{n-1} \\ &= n! \cdot \prod_{i=1}^n t_i^{n-1}. \end{aligned}$$

Thus, $r_G = n! = |W|$. ■

The proof of Corollary 0.0.6 is now anticlimactic:

Proof of Cor. 0.0.6. Combine Theorems 0.0.4 and 0.0.5, and Proposition 5.0.1. ■

6. FINAL REMARKS AND QUESTIONS

We conclude the paper with a discussion of how to generalize Corollary 0.0.6 to arbitrary reductive groups.

Remark 6.0.1. *One can prove that $r_G = |W|$ for any reductive group G over a field k admitting a split maximal torus, but the proof no longer is independent of Martin's theorem (0.0.3).*

We outline a justification of this remark, pointing the reader to [Se] as a reference on geometric invariant theory relative to a base; the base we will use is $\mathbb{Z}_{(p)}$, the localization of \mathbb{Z} at the characteristic p of the base field k .

By Theorems 0.0.4 and 0.0.5, it suffices to verify $r_G^X = |W|$ on a single G -linearized projective variety X for each simple Chevalley group G . Each Chevalley group G admits a model $G_{\mathbb{Z}}$ over the integers, with a split maximal torus $T_{\mathbb{Z}} \subseteq G_{\mathbb{Z}}$. Moreover, there is a smooth projective $\mathbb{Z}_{(p)}$ -scheme $X_{(p)}$ on which $G_{(p)} := G_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{Z}_{(p)}$ acts linearly and for which all $G_{(p)}$ - (resp. $T_{(p)}$ -) semi-stable points are stable and comprise an open locus that nontrivially intersects the closed fibre over \mathbb{F}_p . We justify this assertion briefly: Proposition 2.2.3 reduces the problem to finding some $\mathbb{Z}_{(p)}$ -scheme for which there exist $G_{(p)}$ -stable points in the closed fibre over \mathbb{F}_p ; with the aid of the Hilbert-Mumford criterion, one discovers that many such schemes exist (e.g. take $\mathbb{P}(V_{\mathbb{Z}_{(p)}}^{\oplus n})$ with $V_{\mathbb{Z}_{(p)}}^{\oplus n}$ a large multiple of a general irreducible $G_{(p)}$ -representation).

Having chosen such an $X_{(p)}$, the technique of specialization (cf. [Fu, §20.3]) implies that the integral of relative 0-cycles on $X_{(p)}/G_{(p)}$ and $X_{(p)}/T_{(p)}$ restricted to the generic fibre over \mathbb{Q} is equal to the integral restricted to any closed fibre over \mathbb{F}_p . The ratio r_G is independent under field extension by Lemma 4.1.1, and so this reduces the calculation of r_G over the field k to the computation of $r_{G_{\mathbb{C}}}$, where $G_{\mathbb{C}} := G_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{C}$. The Kirwan-Kempf-Ness theorem (cf. [Ki1, §8] or [FKM, §8.2]) shows that over \mathbb{C} , the GIT quotient $X_{\mathbb{C}}/G_{\mathbb{C}}$ is equivalent to the symplectic reduction, and so Martin's theorem [Ma, Thm. B] proves $r_{G_{\mathbb{C}}} = |W|$.

Question 6.0.2. *What is a purely algebraic proof that $r_G = |W|$ for a general reductive group G admitting a split maximal torus?*

In light of Theorems 0.0.4 and 0.0.5, to answer Question 6.0.2 it suffices to verify $r_G = |W|$ for all simple groups G . Such a verification was done in §5 for simple groups of type \mathbf{A}_n . Can r_G be calculated (algebraically) for any other simple groups G ?

Question 6.0.1. *Are there any interesting combinatorial applications of Corollary 0.0.6?*

The proof of Corollary 0.0.6 boils down to a calculation involving the symmetric group (cf. §5), so perhaps interpreting Corollary 0.0.6 in the context of another explicit example would have interesting combinatorial consequences.

APPENDIX: CHOW GROUPS AND QUOTIENT STACKS

Here we recall the basic properties of Chow groups for schemes and quotient stacks.

A.1. Chow groups. For a scheme X defined over a field k , let $A_i(X)$ denote the \mathbb{Z} -module generated by i -dimensional subvarieties over k modulo rational equivalence (see [Fu]). We call $A_*(X) := \bigoplus_i A_i(X)$ the *Chow group* of X . To indicate rational coefficients, we write $A_*(X)_{\mathbb{Q}} := A_*(X) \otimes \mathbb{Q}$.

For a scheme X over a field k and an algebraic group G acting on X , the Chow group of the quotient stack $[X/G]$ is defined by Edidin and Graham in [EG1] to be the limit of Chow groups using Totaro's [To] finite approximation construction:

$$A_i([X/G]) := A_{i-g+n}(X \times_G U)$$

where U is an open subset of an n -dimensional G -representation V on which G acting freely and whose complement $V \setminus U$ has sufficiently large codimension. It is a result of Edidin and Graham that this is well-defined and independent of the presentation of the stack $[X/G]$ as a quotient (see [EG1, Prop. 16]). Equivalently, we may think of $A_i[X/G]$ as the G -equivariant Chow group of X , and we make use of the notation $A_*^G(X) := A_*([X/G])$ when convenient. The Chow groups of quotient stacks are functorial with respect to the usual operations (e.g. flat pull-back, proper push-forward), and when X is smooth, there is an intersection product that endows these groups with the structure a commutative ring with identity, graded by codimension. Hence the Chow group of the stack $A_*([X/G])$ is naturally a module over the ring $A^*(BG)$ where $BG = [\mathrm{Spec} k/G]$ is the trivial quotient. In the case $T = \mathbb{G}_m^n$ is a rank n torus, we denote

$$S := A^*(BT) = \mathrm{Sym} \Lambda^*(T) \cong \mathbb{Z}[\chi_1, \dots, \chi_n].$$

A character $\chi \in \Lambda^*(T)$ is equivalent to a line bundle L_χ over BT whose Chern class $c_1(L_\chi) = \chi \in S$.

For a G -scheme X , the relationship between the rational Chow groups of $[X/G]$ and $[X/T]$ is simple to state. The following result follows originally from the work of Vistoli [Vi2, Thm. 3.1], but the formulation we require is taken from [Br1]:

Theorem A.1.1. *Let G be a connected reductive group, acting on a k -scheme X , and with maximal torus T . The homomorphism*

$$\gamma : S_{\mathbb{Q}} \otimes_{S_{\mathbb{Q}}^W} A_*^G(X)_{\mathbb{Q}} \rightarrow A_*^T(X)_{\mathbb{Q}}$$

defined by $u \otimes v \mapsto u \frown \pi^(v)$, where $\pi : [X/T] \rightarrow [X/G]$ is the natural surjection, is a W -equivariant isomorphism.*

Proof. See [Br1, Thm. 6.7]. ■

As usual, the case of the action of a torus T is especially well-understood (see [Br1]). In particular, there is a localization theorem useful for making calculations in T -equivariant Chow groups. The following version of the localization theorem will suffice for our purposes:

Theorem A.1.2 (Localization). *Let X be a smooth projective scheme with a T -action, and let $i : X^T \rightarrow X$ denote the inclusion of the scheme of T -fixed points. Then the morphism*

$$i^* : A_*^T(X)_{\mathbb{Q}} \rightarrow A_*^T(X^T)_{\mathbb{Q}}$$

is an injective S -algebra morphism. Furthermore, if X^T consists of finitely many points, then the morphism

$$i^* : A_*^T(X) \rightarrow A_*^T(X^T)$$

of Chow groups with integer coefficients is injective as well.

Proof. See [Br1, Cor. 3.2.1]. ■

A.2. Operational Chow groups. We define the i th operational Chow group $A^i(X)$ to be the group of “operations” c that comprise a system of group homomorphisms $c_f : A_*(Y) \rightarrow A_{*-i}(Y)$, for morphisms of schemes $f : Y \rightarrow X$, compatible with proper push-forward, flat pull-back, and the refined Gysin map (cf. [Fu, §17]). Similarly, Edidin and Graham define equivariant operational Chow groups $A_G^i(X)$ via systems of group homomorphisms $c_f^G : A_*(Y) \rightarrow A_{*-i}^G(Y)$ compatible with the G -equivariant analogues of the above maps (see [EG1, §2.6]). The most obvious examples of equivariant operational Chow classes are equivariant Chern classes $c_i(\mathcal{E})$ of G -linearized vector bundles \mathcal{E} (i.e. Chern classes of vector bundles on $[X/G]$). Moreover, $A_G^*(X)$ equipped with composition forms an associative, graded ring with identity. When X is smooth, there is a Poincaré duality between the equivariant operational Chow group and the usual equivariant Chow group.

Theorem A.2.3 (Poincaré duality). *If X is a smooth n -dimensional variety, then the map $A_G^i(X) \rightarrow A_{n-i}^G(X)$ defined by $c \mapsto c \frown [X]$ is an isomorphism.*

Proof. See [EG1, Prop. 4]. ■

Remark A.2.4. *When X is a smooth n -dimensional variety, this allows us to write $A_G^k(X)$ to denote the codimension k Chow group $A_{n-k}^G(X)$. Furthermore, this induces an isomorphism of rings $A_G^*(X) \cong A_{n-*}^G(X)$, with the multiplication structure on $A_{n-*}^G(X)$ given by the intersection product.*

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